The Exp-Function Method for the Riccati Equation and Exact Solutions of Dispersive Long Wave Equations

Sheng Zhang, Wei Wang, and Jing-Lin Tong

Department of Mathematics, Bohai University, Jinzhou 121000, P. R. China

Reprint requests to S. Z.; E-mail: dr.szhang@yahoo.com.cn

Z. Naturforsch. **63a,** 663 – 670 (2008); received April 23, 2008

In this paper, the Exp-function method is used to seek new generalized solitonary solutions of the Riccati equation. Based on the Riccati equation and one of its generalized solitonary solutions, new exact solutions with three arbitrary functions of the (2+1)-dimensional dispersive long wave equations are obtained. Compared with the tanh-function method and its extensions, the proposed method is more powerful. It is shown that the Exp-function method provides a straightforward and important mathematical tool for solving nonlinear evolution equations in mathematical physics.

Key words: Exp-Function Method; Riccati Equation; Tanh-Function Method; Generalized Solitonary Solutions; Nonlinear Evolution Equations.

1. Introduction

It is well known that nonlinear evolution equations (NLEEs) are often presented to describe the motion of isolated waves, localized in a small part of space, in many fields such as hydrodynamics, plasma physics, nonlinear optics. The investigation of exact solutions of NLEEs plays an important role in the study of these nonlinear physical phenomena. With the development of the soliton theory, many significant methods for obtaining exact solutions of NLEEs have been presented such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine-cosine method [5], homogenous balance method [6], homotopy perturbation method [7 – 9], variational method [10 – 13], asymptotic methods [14, 15], Adomian decomposition method [16], mapping approach [17], algebraic method [18 – 21], Jacobi elliptic function expansion method [22 – 24], F-expansion method [25 – 30], auxiliary equation method [31 – 35]. One of the most effective methods to construct solitary wave solutions of NLEEs is the tanh-function method [36, 37]. The method was later extended by Ma [38], Gao and Tian [39], and Fan [40]. Elwakil et al. [41] extended Fan's method [40], and Lü and Zhang [42] further extended Elwakil et al.'s work [41]. By generalizing the Riccati equation and introducing its special solutions, the method [42] was further improved in [43 – 46].

Generally speaking, exact solutions of NLEEs obtained by most of these methods are written as a polynomial in several elementary or special functions which satisfy a first-order ordinary differential equation called sub-equation, for example, sine-Gordon equation, elliptic equation, Riccati equation and others. It is obvious that the more solutions of these sub-equations we find, the more exact solutions of the considered NLEEs we may obtain.

Recently, He and Wu [47] proposed the Exp-function method to obtain exact solutions of NLEEs. The basic idea of the Exp-function method can be found in He's monograph [48]. The solution procedure of the Exp-function method, by the help of Matlab or Mathematica, is of utter simplicity and it has been successfully applied to many kinds of NLEEs [47 – 63]. The main disadvantage of the Adomian decomposition method is the difficulty in calculating the so-called Adomian polynomial [50]. The Exp-function method leads to both generalized solitonary solutions and periodic solutions. Taking advantage of the generalized solitonary solutions with free parameters, some known solutions gained by the most existing methods such as Adomian decomposition method, tanh-function method, algebraic method, Jacobi elliptic function expansion method, F-expansion method, and auxiliary equation method can be recovered as special cases [55]. The classical Jacobi elliptic function expansion method, tanh-function method and F-expansion method can't be used to solve the NLEEs in which the odd- and even-order derivative terms coexist [60]. However, Ebaid [60] took the Burgers equation as an example to illustrate that the Exp-function method is valid for such NLEEs. Furthermore, the Exp-function method for discrete NLEEs [63] is more powerful than the hyperbolic function method [64].

The present paper is motivated by the desire to use the Exp-function method to seek new generalized solitonary solutions of the Riccati equation [42]

$$\phi'(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi) = \delta + \phi^2(\xi), \tag{1}$$

and to employ the Riccati equation (1) and its generalized solitonary solutions, and the method [42] to find new and more general exact solutions of the (2+1)-dimensional dispersive long wave (DLW) equations

$$u_{yt} + H_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \tag{2}$$

$$H_t + (uH + u + u_{xy})_x = 0. \tag{3}$$

Equations (2) and (3) were firstly obtained by Boiti et al. [65] as a compatibility condition for a "weak" Lax pair. Paquin and Winternitz [66] showed that the symmetry algebra of (2) and (3) is infinite-dimensional. The more general symmetry algebra, W_∞ symmetry algebra, was given in [67]. Lou [68] gave nine types of two-dimensional similarity reductions and thirteen types of ordinary differential equation reductions. He [13] established a variational model of (2) and (3) by using the semi-inverse method.

2. The Exp-Function Method for the Riccati Equation

Introducing a complex variable η defined as

$$\eta = k(\xi + \omega), \tag{4}$$

where k is a constant to be determined later and ω is an arbitrary constant, (1) becomes

$$k\phi' - \delta - \phi^2 = 0, \tag{5}$$

where the prime denotes the derivative with respect to η.

According to the Exp-function method [47], we assume that the solution of (5) can be expressed as

$$\phi(\eta) = \frac{a_c\exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_f\exp(f\eta) + \cdots + b_{-g}\exp(-g\eta)}, \tag{6}$$

where c, d, f and g are positive integers, which are unknown, to be further determined, and $a_c, \ldots, a_{-d}$, $b_f, \ldots, b_{-g}$ are unknown constants.

In order to determine values of c and f, we balance the linear term of highest-order in (5) with the highest-order nonlinear term [47]. By simple calculation, we have

$$\phi' = \frac{c_1\exp[(f+c)\eta] + \cdots}{c_2\exp(2f\eta) + \cdots} \tag{7}$$

and

$$\phi^2 = \frac{c_3\exp(2c\eta) + \cdots}{c_4\exp(2f\eta) + \cdots}, \tag{8}$$

where c_i are determined coefficients only for simplicity.

Balancing the highest-order of the Exp-function in (7) and (8), we have

$$f + c = 2c, \tag{9}$$

which leads to the result $f = c$.

Similarly to determine values of d and g, we balance the linear term of lowest-order in (5):

$$\phi' = \frac{\cdots + d_1\exp[-(g+d)\eta]}{\cdots + d_2\exp(-2g\eta)} \tag{10}$$

and

$$\phi^2 = \frac{\cdots + d_3\exp(-2d\eta)}{\cdots + d_4\exp(-2g\eta)}, \tag{11}$$

where d_i are determined coefficients only for simplicity.

Balancing the lowest-order of the Exp-function in (10) and (11), we have

$$-(g+d) = -2d, \tag{12}$$

which leads to the result $g = d$.

We can freely choose the values of c and d, but the final solution does not strongly depend upon the choice of the values of c and d [47,49]. For simplicity, we set $f = c = 1$ and $d = g = 1$; then (6) becomes

$$\phi(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{13}$$

Substituting (13) into (5), we have

$$\frac{1}{A}\left[C_2\exp(2\eta) + C_1\exp(\eta) + C_0 + C_{-1}\exp(-\eta) + C_{-2}\exp(-2\eta)\right] = 0, \tag{14}$$

where

$$
\begin{aligned}
&A = [b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)]^2,\\
&C_2 = -a_{-1}^2 - b_{-1}^2 \delta,\\
&C_1 = -2a_0 a_{-1} - a_{-1} b_0 k + a_0 b_{-1} k - 2b_0 b_{-1} \delta,\\
&C_0 = -a_0^2 - 2a_1 a_{-1} - 2a_{-1} b_1 k\\
&\qquad + 2a_1 b_{-1} k - b_0^2 \delta - 2b_1 b_{-1} \delta,\\
&C_{-1} = -2a_0 a_1 + a_1 b_0 k - a_0 b_1 k - 2b_0 b_1 \delta,\\
&C_{-2} = -a_1^2 - b_1^2 \delta.
\end{aligned}
$$

Equating the coefficients of $\exp(j\eta)$ $(j = 2, 1, 0, -1, -2)$ to zero, we have

$$
\begin{cases}
C_2 = 0, \quad C_1 = 0,\\
C_0 = 0,\\
C_{-2} = 0, \quad C_{-1} = 0.
\end{cases}
$$

Solving this system of algebraic equations with the aid of Mathematica, we obtain

$$a_1 = \pm b_1 \sqrt{-\delta}, \quad a_0 = 0, \quad a_{-1} = \mp b_{-1} \sqrt{-\delta}, \tag{15}$$

$$b_1 = b_1, \quad b_0 = 0, \quad b_{-1} = b_{-1}, \quad k = \mp \sqrt{-\delta}, \tag{16}$$

or

$$
\begin{aligned}
&a_1 = \pm b_1 \sqrt{-\delta}, \quad a_0 = a_0,\\
&a_{-1} = \mp \frac{(a_0^2 + b_0^2 \delta)\sqrt{-\delta}}{4b_1 \delta},
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&b_1 = b_1, \quad b_0 = b_0, \quad b_{-1} = \frac{a_0^2 + b_0^2 \delta}{4b_1 \delta},\\
&k = \mp 2\sqrt{-\delta}.
\end{aligned} \tag{18}
$$

Substituting (15) and (16) into (13), we obtain the following generalized solitonary solution of (1):

$$
\begin{aligned}
\phi = &\big\{ \pm b_1 \sqrt{-\delta} \exp[\mp \sqrt{-\delta}(\xi + \omega)]\\
&\mp b_{-1} \sqrt{-\delta} \exp[\pm \sqrt{-\delta}(\xi + \omega)]\big\}\\
&\cdot \big\{ b_1 \exp[\mp \sqrt{-\delta}(\xi + \omega)]\\
&+ b_{-1} \exp[\pm \sqrt{-\delta}(\xi + \omega)]\big\}^{-1},
\end{aligned} \tag{19}
$$

which has not been given in [42] yet.

When $\frac{b_{-1}}{b_1} > 0$ and $\delta < 0$, we set $\omega = \mp \frac{1}{2\sqrt{-\delta}} \ln \frac{b_{-1}}{b_1}$. Then (19) reduces to the first hyperbolic function solution, given by Lü and Zhang in [42],

$$
\begin{aligned}
\phi &= \pm \sqrt{-\delta} \frac{\exp(\mp \sqrt{-\delta}\xi) - \exp(\pm \sqrt{-\delta}\xi)}{\exp(\mp \sqrt{-\delta}\xi) + \exp(\pm \sqrt{-\delta}\xi)}\\
&= -\sqrt{-\delta} \tanh(\sqrt{-\delta}\xi),
\end{aligned} \tag{20}
$$

which shows that solutions (19) and (20) have the same waveform and are different in phase.

When $\frac{b_{-1}}{b_1} < 0$ and $\delta < 0$, we set $\omega = \mp \frac{1}{2\sqrt{-\delta}} \ln(-\frac{b_{-1}}{b_1})$. Then (19) reduces to the second hyperbolic function solution, given by Lü and Zhang in [42],

$$
\begin{aligned}
\phi &= \pm \sqrt{-\delta} \frac{\exp(\mp \sqrt{-\delta}\xi) + \exp(\pm \sqrt{-\delta}\xi)}{\exp(\mp \sqrt{-\delta}\xi) - \exp(\pm \sqrt{-\delta}\xi)}\\
&= -\sqrt{-\delta} \coth(\sqrt{-\delta}\xi),
\end{aligned} \tag{21}
$$

which shows that solutions (19) and (21) have also the same waveform and are different in phase.

When $\frac{b_{-1}}{b_1}$ is an appropriate complex number and $\delta < 0$, however, some new solutions can be obtained from solution (19). For example, setting $b_1 = 1$ and $b_{-1} = -(1 \pm \mathrm{i})$, here and thereafter $\mathrm{i} \equiv \sqrt{-1}$, we obtain

$$
\begin{aligned}
\phi = &\pm \sqrt{-\delta} \big\{ \exp[\mp \sqrt{-\delta}(\xi + \omega)]\\
&+ (1 \pm \mathrm{i}) \exp[\pm \sqrt{-\delta}(\xi + \omega)]\big\}\\
&\cdot \big\{ \exp[\mp \sqrt{-\delta}(\xi + \omega)]\\
&- (1 \pm \mathrm{i}) \exp[\pm \sqrt{-\delta}(\xi + \omega)]\big\}^{-1},
\end{aligned} \tag{22}
$$

which can be changed into

$$
\begin{aligned}
\phi = &\pm \sqrt{-\delta} \big\{ \exp[\mp 2\sqrt{-\delta}(\xi + \omega)] \pm 2\mathrm{i}\\
&- 2\exp[\pm 2\sqrt{-\delta}(\xi + \omega)]\big\}\\
&\cdot \big\{ \exp[\mp 2\sqrt{-\delta}(\xi + \omega)] - 2\\
&+ 2\exp[\pm 2\sqrt{-\delta}(\xi + \omega)]\big\}^{-1}.
\end{aligned} \tag{23}
$$

Further setting $\omega = \mp \frac{1}{4\sqrt{-\delta}} \ln 2 + \varpi$, ϖ is an arbitrary constant, then (23) can be reduced to

$$
\begin{aligned}
\phi = &-\sqrt{-\delta} \big\{ \sqrt{2} \tanh[2\sqrt{-\delta}(\xi + \varpi)]\\
&- \mathrm{i}\,\mathrm{sech}[2\sqrt{-\delta}(\xi + \varpi)]\big\}\\
&\cdot \big\{ \sqrt{2} - \mathrm{sech}[2\sqrt{-\delta}(\xi + \varpi)]\big\}^{-1}.
\end{aligned} \tag{24}
$$

In Fig. 1, a plot of the real part of solution (24) with $\delta = -1$ and $\varpi = 0$ is shown in the interval $\xi \in [-\pi, \pi]$.

When $b_1 = 1$, $b_{-1} = 1$, $\omega = 0$ and $\delta > 0$, then (19) becomes

$$
\begin{aligned}
\phi &= \pm \mathrm{i}\sqrt{\delta} \frac{\exp(\mp \mathrm{i}\sqrt{\delta}\xi) - \exp(\pm \mathrm{i}\sqrt{\delta}\xi)}{\exp(\mp \mathrm{i}\sqrt{\delta}\xi) + \exp(\pm \mathrm{i}\sqrt{\delta}\xi)}\\
&= \sqrt{\delta} \tan(\sqrt{\delta}\xi),
\end{aligned} \tag{25}
$$

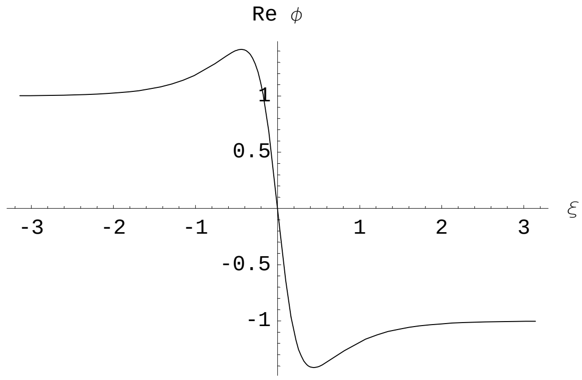

Fig. 1. A plot of the real part of solution (24).

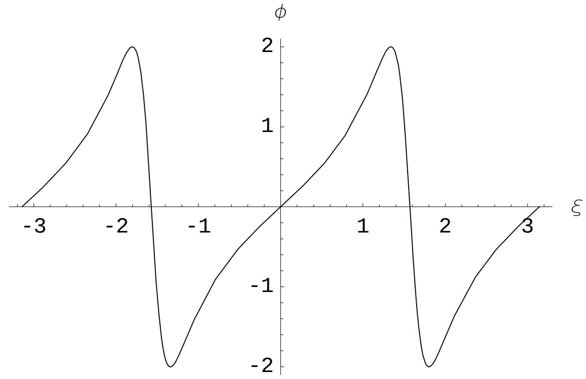

Fig. 2. A plot of the real part of solution (28).

which is the first triangular function solution given by Lü and Zhang in [42].

When $b_1=1$, $b_{-1}=-1$, $\omega=0$ and $\delta>0$, then (19) becomes

$$\phi=\pm\mathrm{i}\sqrt{\delta}\frac{\exp(\mp\mathrm{i}\sqrt{\delta}\xi)+\exp(\pm\mathrm{i}\sqrt{\delta}\xi)}{\exp(\mp\mathrm{i}\sqrt{\delta}\xi)-\exp(\pm\mathrm{i}\sqrt{\delta}\xi)}=-\sqrt{\delta}\cot(\sqrt{\delta}\xi),\tag{26}$$

which is the second triangular function solution given by Lü and Zhang in [42].

When $\frac{b_{-1}}{b_1}\neq\pm1$ and $\delta>0$, some new solutions can be obtained from solution (19). For example, setting $b_1=1$ and $b_{-1}=\frac{\sqrt{5}\mp1}{2}$, we obtain

$$\begin{aligned}\phi=&\pm\mathrm{i}\sqrt{\delta}\big\{\exp[\mp\mathrm{i}\sqrt{\delta}(\xi+\omega)]-\frac{\sqrt{5}\mp1}{2}\exp[\pm\mathrm{i}\sqrt{\delta}(\xi+\omega)]\big\}\\&\cdot\big\{\exp[\mp\mathrm{i}\sqrt{\delta}(\xi+\omega)]+\frac{\sqrt{5}\mp1}{2}\exp[\pm\mathrm{i}\sqrt{\delta}(\xi+\omega)]\big\}^{-1},\end{aligned}\tag{27}$$

which can be converted into

$$\phi=\sqrt{\delta}\frac{2\tan[2\sqrt{\delta}(\xi+\omega)]+\mathrm{i}\sec[2\sqrt{\delta}(\xi+\omega)]}{2+\sqrt{5}\sec[2\sqrt{\delta}(\xi+\omega)]}.\tag{28}$$

In Fig. 2, a plot of the real part of solution (28) with $\delta=1$ and $\omega=0$ is shown in the interval $\xi\in[-\pi,\pi]$.

From (17) and (18), we obtain the following generalized solitonary solution of (1):

$$\begin{aligned}\phi=&\big\{\pm b_1\sqrt{-\delta}\exp[\mp2\sqrt{-\delta}(\xi+\omega)]+a_0\\&\mp\frac{(a_0^2+b_0^2\delta)}{4b_1\delta}\sqrt{-\delta}\exp[\pm2\sqrt{-\delta}(\xi+\omega)]\big\}\\&\cdot\big\{b_1\exp[\mp2\sqrt{-\delta}(\xi+\omega)]+b_0\\&+\frac{a_0^2+b_0^2\delta}{4b_1\delta}\exp[\pm2\sqrt{-\delta}(\xi+\omega)]\big\}^{-1}.\end{aligned}\tag{29}$$

Equation (29) can be simplified in the form

$$\begin{aligned}\phi=&\big\{\pm b_1\sqrt{-\delta}\exp[\mp\sqrt{-\delta}(\xi+\omega)]\\&\mp\frac{b_0\sqrt{-\delta}\mp a_0}{2}\exp[\pm\sqrt{-\delta}(\xi+\omega)]\big\}\\&\cdot\big\{b_1\exp[\mp\sqrt{-\delta}(\xi+\omega)]\\&+\frac{b_0\sqrt{-\delta}\mp a_0}{2\sqrt{-\delta}}\exp[\pm\sqrt{-\delta}(\xi+\omega)]\big\}^{-1},\end{aligned}\tag{30}$$

which shows that solution (29) is equal to solution (19) because of the arbitrariness of a_0, b_0 and b_{-1}.

Here we would like to point out that when $\delta=0$, (1) also has the rational solution

$$\phi=-\frac{1}{\xi+\omega},\tag{31}$$

which converts into the one given by Lü and Zhang in [42], if setting $\omega=0$.

There are no other solutions of (1) given in [42] except the ones mentioned above. To compare solution (19) with the known ones, four plots of solutions (20), (21), (25), and (26) are shown in Figs. 3 – 6; in the interval $\xi\in[-\pi,\pi]$. Figures 1 – 6 show that some new solutions with different waveforms can be

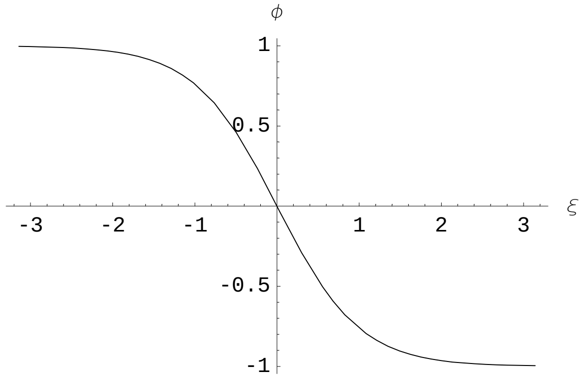

Fig. 3. A plot of solution (20) with $\delta = -1$.

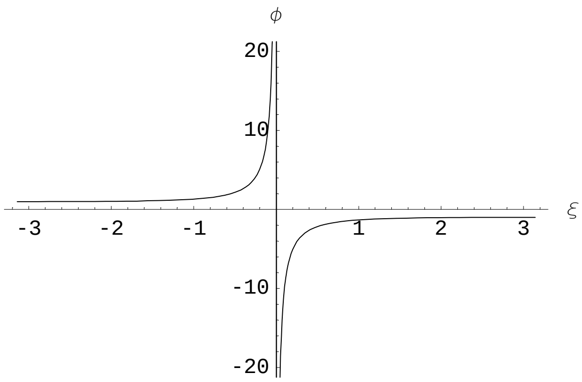

Fig. 4. A plot of solution (21) with $\delta = -1$.

obtained from solution (19), i. e. solution (19) includes not only all known solutions given in [42] as special cases, but also some new ones. More important, if we use the method proposed in [42] and solution (19), we can obtain new and more general exact solutions including all solutions obtained by the method [42] as special cases. In the next section, we will use the Riccati equation (1) and its general solitonary solution (19), combined with method [42], to construct new and more general solutions of (2) and (3).

3. Exact Solutions of the (2+1)-Dimensional DLW Equations

Balancing the highest-order partial derivative with the nonlinear term [42], we suppose that (2) and (3) have the following formal solutions:

$$u = a_0(y,t) + a_{-1}(y,t)\phi^{-1}(\xi) + a_1(y,t)\phi(\xi), \quad (32)$$

$$H = b_0(y,t) + b_{-1}(y,t)\phi^{-1}(\xi) + b_{-2}(y,t)\phi^{-2}(\xi) + b_1(y,t)\phi(\xi) + b_2(y,t)\phi^2(\xi), \quad (33)$$

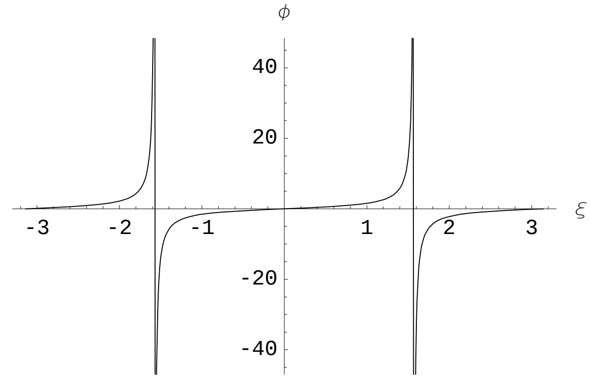

Fig. 5. A plot of solution (25) with $\delta = 1$.

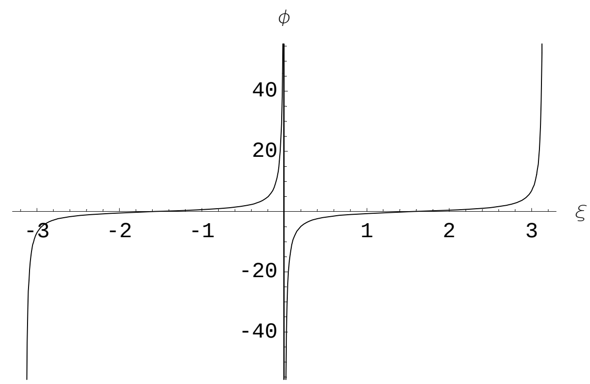

Fig. 6. A plot of solution (26) with $\delta = 1$.

where $\phi(\xi)$ satisfies the Riccati equation (1), $\xi = kx + \eta(y,t)$, $a_0(y,t)$, $a_{-1}(y,t)$, $a_1(y,t)$, $b_0(y,t)$, $b_{-1}(y,t)$, $b_{-2}(y,t)$, $b_1(y,t)$, $b_2(y,t)$ and $\eta(y,t)$ are functions of y and t to be determined later, k is a nonzero constant.

Substituting (32) and (33) along with (1) into (2) and (3), the left-hand sides of (2) and (3) are converted into two polynomials of $\phi^i(\xi)$ $(i = 0, \pm 1, \pm 2, \pm 3, \pm 4)$. Then setting each coefficient to zero, we get a set of over-determined partial differential equations for $a_0(y,t)$, $a_{-1}(y,t)$, $a_1(y,t)$, $b_0(y,t)$, $b_{-1}(y,t)$, $b_{-2}(y,t)$, $b_1(y,t)$, $b_2(y,t)$ and $\eta(y,t)$. Solving the set of over-determined partial differential equations by the use of Mathematica, we get the following four cases:

Case 1.

$$a_0(y,t) = -\frac{f(y) + h'(t)}{k}, \quad a_{-1}(y,t) = \pm 2k\delta, \quad a_1(y,t) = \mp 2k,$$

$$b_0(y,t) = -1 \mp \frac{f'(y)}{k} - 4k\delta[tf'(y) + g'(y)], \quad b_{-1}(y,t) = 0, \quad b_{-2}(y,t) = -2k\delta^2[tf'(y) + g'(y)],$$

$$b_1(y,t)=0,\quad b_2(y,t)=-2k[tf'(y)+g'(y)],$$
$$\eta(y,t)=tf(y)+g(y)+h(t),$$

where $f(y)$, $g(y)$ and $h(t)$ are arbitrary functions of the indicated valuables, respectively, $f'(y)=\mathrm{d}f(y)/\mathrm{d}y$, $g'(y)=\mathrm{d}g(y)/\mathrm{d}y$, $h'(t)=\mathrm{d}h(t)/\mathrm{d}t$.

Case 2.

$$a_0(y,t)=-\frac{f(y)+h'(t)}{k},\quad a_{-1}(y,t)=\pm 2k\delta,$$
$$a_1(y,t)=0,$$

$$b_0(y,t)=-1\mp\frac{f'(y)}{k}-2k\delta[tf'(y)+g'(y)],$$
$$b_{-1}(y,t)=0,$$
$$b_{-2}(y,t)=-2k\delta^2[tf'(y)+g'(y)],$$

$$b_1(y,t)=0,\quad b_2(y,t)=0,$$
$$\eta(y,t)=tf(y)+g(y)+h(t),$$

where $f(y)$, $g(y)$ and $h(t)$ are arbitrary functions of the indicated valuables, respectively, $f'(y)=\mathrm{d}f(y)/\mathrm{d}y$, $g'(y)=\mathrm{d}g(y)/\mathrm{d}y$, $h'(t)=\mathrm{d}h(t)/\mathrm{d}t$.

Case 3.

$$a_0(y,t)=-\frac{f(y)+h'(t)}{k},\quad a_{-1}(y,t)=0,$$
$$a_1(y,t)=\pm 2k,$$

$$b_0(y,t)=-1\pm\frac{f'(y)}{k}-2k\delta[tf'(y)+g'(y)],$$
$$b_{-1}(y,t)=0,\quad b_{-2}(y,t)=0,$$

$$b_1(y,t)=0,\quad b_2(y,t)=-2k[tf'(y)+g'(y)],$$
$$\eta(y,t)=tf(y)+g(y)+h(t),$$

where $f(y)$, $g(y)$ and $h(t)$ are arbitrary functions of the indicated valuables, respectively, $f'(y)=\mathrm{d}f(y)/\mathrm{d}y$, $g'(y)=\mathrm{d}g(y)/\mathrm{d}y$, $h'(t)=\mathrm{d}h(t)/\mathrm{d}t$.

Case 4.

$$a_0(y,t)=-\frac{h'(t)}{k},\quad a_{-1}(y,t)=\pm 2k\delta,$$
$$a_1(y,t)=\pm 2k,$$

$$b_0(y,t)=-1,\quad b_{-1}(y,t)=0,$$
$$b_{-2}(y,t)=-2k\delta^2 g'(y),$$

$$b_1(y,t)=0,\quad b_2(y,t)=-2kg'(y),$$
$$\eta(y,t)=g(y)+h(t),$$

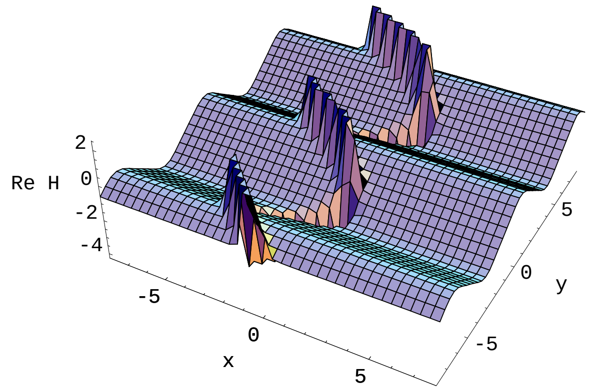

Fig. 7. A plot of the real part of solution (35).

where $g(y)$ and $h(t)$ are arbitrary functions of the indicated valuables, respectively, $g'(y)=\mathrm{d}g(y)/\mathrm{d}y$, $h'(t)=\mathrm{d}h(t)/\mathrm{d}t$.

From (19), (32) – (36) we obtain the following formal exact solutions of (2) and (3):

$$u=-\frac{f(y)+h'(t)}{k}\pm 2k\delta\phi^{-1}(\xi)\mp 2k\phi(\xi), \quad (34)$$

$$\begin{aligned}H=&-1\mp\frac{f'(y)}{k}-4k\delta[tf'(y)+g'(y)]\\&-2k\delta^2[tf'(y)+g'(y)]\phi^{-2}(\xi)\\&-2k[tf'(y)+g'(y)]\phi^2(\xi),\end{aligned}\quad (35)$$

where $\phi(\xi)$ is determined by (19) with $\xi=kx+tf(y)+g(y)+h(t)$.

The free parameters b_1, a_0, b_0 and the arbitrary functions $f(y)$, $g(y)$ and $h(t)$ in solutions (34) and (35) let us to discuss the behaviours of the solutions and also provided us with enough freedom to construct solutions that may be related to real physical problems. If we choose $b_1=1$, $b_{-1}=-\frac{1+\mathrm{i}}{2}$, $\omega=0$, $f(y)=\mathrm{sn}(y|0.3)$, $g(y)=\mathrm{cn}(y|0.5)$, $h(t)=\sin(t)$ and $t=\frac{\pi}{2}$, then a new spatial structure of the real part of solution (35) is shown in the region $\{x\in[-8,8], y\in[-8,8]\}$ in Fig. 7, from which we can see that solution (35) possesses solitonic features.

From Cases 2 – 4 and (19), we can also obtain other exact solutions of (2) and (3); we do not consider them here for simplicity. Solutions (34) and (35) and the omitted ones can not be obtained by the tanh-function method [36, 37] and its extensions [38 – 46]. To the best of our knowledge, these solutions have not been reported in the literature.

Remark: With the aid of Mathematica, we have checked the obtained solutions by putting them back into the original equations (2) and (3).

4. Conclusion

The Exp-function method has been successfully used to seek generalized solitonary solutions of the Riccati equation. Based on the Riccati equation and its generalized solitonary solutions, new exact solutions with three arbitrary functions of the (2+1)-dimensional DLW equations are obtained. The arbitrary functions imply that these solutions have rich spatial structures. It may be important to explain some physical phenomena. Compared with the tanh-function method [36, 37] and its extensions [38 – 46], the method proposed in this paper gives not only more general solutions but also new formal solutions. Taking full advantage of the free parameters b_1 and b_{-1}, solution (19) can be used to improve the methods, in which (1) is used as sub-equation, for finding new and interesting solutions hidden in NLEEs. It shows that the Exp-function method provides a straightforward and important mathematical tool for NLEEs in mathematical physics. Applications of the Exp-function method to other sub-equations are worth to be further studied.

Acknowledgements

This work was supported by the Natural Science Foundation of Educational Committee of Liaoning Province of China (Grant No. 20060022).

[1] M. J. Ablowitz and P. A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge 1991.
[2] R. Hirota, Phys. Rev. Lett. **27**, 1192 (1971).
[3] M. R. Miurs, Bäcklund Transformation, Springer, Berlin 1978.
[4] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).
[5] C. T. Yan, Phys. Lett. A **224**, 77 (1996).
[6] M. L. Wang, Phys. Lett. A **213**, 279 (1996).
[7] M. El-Shahed, Int. J. Nonlinear Sci. Numer. Simul. **6**, 163 (2005).
[8] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[9] J. H. He, Chaos, Solitons and Fractals **26**, 695 (2005).
[10] J. H. He, Int. J. Nonlinear Mech. **34**, 699 (1999).
[11] J. H. He, Appl. Math. Comput. **114**, 115 (2000).
[12] J. H. He, Chaos, Solitons and Fractals **19**, 847 (2004).
[13] J. H. He, Phys. Lett. A **335**, 182 (2005).
[14] J. H. He, Int. J. Modern Phys. B **20**, 1141 (2006).
[15] J. H. He, Int. J. Modern Phys. B **20**, 2561 (2006).
[16] T. A. Abassy, M. A. El-Tawil, and H. K. Saleh, Int. J. Nonlinear Sci. Numer. Simul. **5**, 327 (2004).
[17] S. H. Ma, J. P. Fang, and C. L. Zheng, Z. Naturforsch. **62a**, 8 (2007).
[18] Y. Chen, Q. Wang, and Y. Lang, Z. Naturforsch. **60a**, 127 (2005).
[19] E. Yomba, Chaos, Solitons and Fractals **27**, 187 (2006).
[20] S. Zhang and T. C. Xia, Phys. Lett. A **356**, 119 (2006).
[21] S. Zhang and T. C. Xia, Appl. Math. Comput. **182**, 1651 (2006).
[22] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, Phys. Lett. A **289**, 69 (2001).
[23] E. J. Parkes, B. R. Duffy, and P. C. Abbott, Phys. Lett. A **295**, 280 (2002).
[24] Y. Chen, Q. Wang, and B. Li, Z. Naturforsch. **59a**, 529 (2004).
[25] S. Zhang, Phys. Lett. A **358**, 414 (2006).
[26] S. Zhang, Chaos, Solitons and Fractals **30**, 1213 (2006).
[27] S. Zhang, Chaos, Solitons and Fractals **32**, 847 (2007).
[28] S. Zhang, Chaos, Solitons and Fractals **32**, 1375 (2007).
[29] S. Zhang and T. C. Xia, Appl. Math. Comput. **183**, 1190 (2006).
[30] S. Zhang, Appl. Math. Comput. **189**, 836 (2007).
[31] S. Zhang and T. C. Xia, J. Phys. A: Math. Theor. **40**, 227 (2007).
[32] S. Zhang and T. C. Xia, Phys. Lett. A **363**, 356 (2007).
[33] S. Zhang, Appl. Math. Comput. **188**, 1 (2007).
[34] S. Zhang, Z. Naturforsch. **62a**, 689 (2007).
[35] S. Zhang, Comput. Math. Appl. **54**, 1028 (2007).
[36] L. Huibin and W. Kelin, J. Phys. A: Math. Gen. **23**, 4097 (1990).
[37] W. Malfliet, Am. J. Phys. **60**, 650 (1992).
[38] W. X. Ma, Int. J. Nonlinear Mech. **31**, 329 (1996).
[39] Y. T. Gao and B. Tian, Comput. Math. Appl. **33**, 115 (1997).
[40] E. G. Fan, Phys. Lett. A **227**, 212 (2000).
[41] S. A. Elwakil, S. K. El-labany, M. A. Zahran, and R. Sabry, Phys. Lett. A **299**, 179 (2002).
[42] Z. S. Lü and H. Q. Zhang, Commun. Theor. Phys. (Beijing, China) **39**, 405 (2003).
[43] F. D. Xie, Y. Zhang, and Z. S. Lü, Chaos, Solitons and Fractals **24**, 257 (2005).
[44] S. Zhang and T. C. Xia, Commun. Theor. Phys. (Beijing, China) **45**, 985 (2006).
[45] S. Zhang and T. C. Xia, Appl. Math. Comput. **181**, 319 (2006).
[46] S. Zhang, Chaos, Solitons and Fractals **31**, 951 (2007).

[47] J. H. He and X. H. Wu, Chaos, Solitons and Fractals **30**, 700 (2006).
[48] J. H. He, Non-Perturbative Methods for Strongly Nonlinear Problems, Dissertation, de-Verlag im Internet GmbH, Berlin 2006.
[49] J. H. He and M. A. Abdou, Chaos, Solitons and Fractals **34**, 1421 (2007).
[50] J.H. He and L.N. Zhang, Phys. Lett. A **372**, 1044 (2008).
[51] X. H. Wu and J. H. He, Chaos, Solitons and Fractals **38**, 903 (2008).
[52] X.H. Wu and J. H. He, Comput. Math. Appl. **54**, 966 (2007).
[53] M.A. Abdou, A.A. Soliman, and S.T. El-Basyony, Phys. Lett. A **369**, 469 (2007).
[54] S.A. El-Wakil, M.A. Madkour, and M.A. Abdou, Phys. Lett. A **369**, 62 (2007).
[55] S. Zhang, Phys. Lett. A **365**, 448 (2007).
[56] S. Zhang, Phys. Lett. A **371**, 65 (2007).
[57] S. Zhang, Chaos, Solitons and Fractals **38**, 270 (2008).
[58] S. Zhang, Nonlinear Dyn. **52**, 11 (2008).
[59] S. Zhang, Appl. Math. Comput. **197**, 128 (2008).
[60] A. Ebaid, Phys. Lett. A **365**, 213 (2007).
[61] S. D. Zhu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 461 (2007).
[62] S. D. Zhu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 465 (2007).
[63] S. D. Zhu, Phys. Lett. A **372**, 654 (2008).
[64] J. M. Zhu, Chin. Phys. **14**, 1290 (2005).
[65] M. Boiti, J. J. P. Leon, M. Manna, and F. Pempinelli, Inverse Problem **3**, 25 (1987).
[66] G. Paquin and P. Winternitz, Physica D **46**, 122 (1990).
[67] S. Y. Lou, J. Phys. A: Math. Gen. **27**, 3235 (1994).
[68] S. Y. Lou, Math. Meth. Appl. Sci. **18**, 789 (1995).